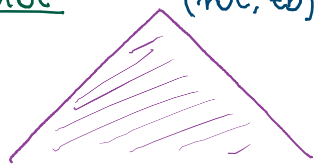
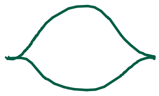
LEGENDRIAN CLASSIFICATION RESULTS

Unknot: $(rot, tb) = (0, -1)$ [Bennequin bound: $tb + |rot| \leq -\chi(disc) = -1$]

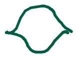




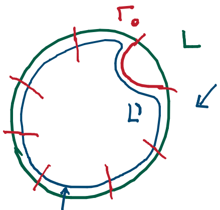
Legendrian unknot

Need to prove: L Leg. repr. of the unknot

A if $tb(L) < -1 \Rightarrow L$ destabilises

B if $tb(L) = -1 \Rightarrow L$ Legendrian isotopic to  L_0

A: 1 Take a smooth disc bounded by L.
2 Since $tb(L) \leq 0 \Rightarrow$ the disc can be C^∞-perturbed rel ∂ to be convex: D

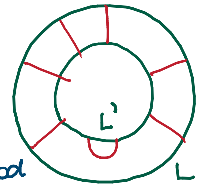


$|L \cap \Gamma| = -2\,tb(L) \geq 4$

3 take an outermost dividing curve: Γ_0
($D - \Gamma_0$ has a component disjoint from Γ)

4 by the Legendrian realisation principle (both component of $D - L'$ contain Γ)
D has a C^∞-perturbation rel ∂
s.t. L' is Legendrian

Then L' is a stabilisation of L (cobounds an annulus with L with dividing curves:

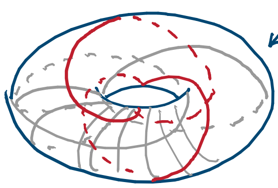


Q: Where did we use $tb < -1$?

B: Suppose $tb(L) = -1$: We will prove that $S^3 - N(L)$ is contactomorphic to $S^3 - N(L_0)$

standard contact neighborhood

$V = S^3 - N(L)$ is a solid torus (the "outside")



meridional foliation (ruling curves)

singular curves giving the Thurston-Bennequin framing $sl = -1$ (Legendrian divides)

Γ dividing curves with $sl = -1$

$\mathcal{F}$

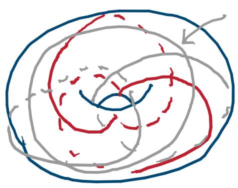
enough to prove that there is a unique tight contact structure on the solid torus that 'prints' $\mathcal{F}$ to its boundary torus

Principle: Only the dividing curve matters:

M closed manifold with boundary Σ and characteristic foliations $\mathcal{F}$ & $\mathcal{F}'$ divided by the same dividing curve Γ. Then there is a 1-1 correspondence between contact structures that "print" $\mathcal{F}$ or $\mathcal{F}'$ on Σ.

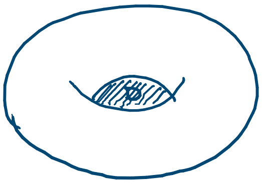
[proof is by Giroux's flexibility Thm]

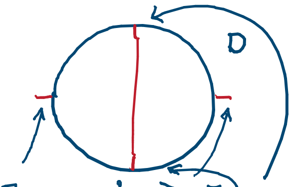
Change the foliation to



ruling is by longitudinal curves

and take the disk with Legendrian boundary: C^∞-perturb D rel ∂ to be convex.

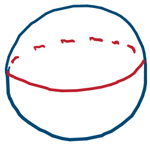




Γ on T^2 $\Rightarrow$ Γ on D } and since $(\mathbb{R}^3, \xi_{st})$ is tight the dividing curve must be like this

Now after smoothing $V - N(D)$ is a ball with a 1-component dividing curve on it

[don't need to check since $\subseteq \xi_{st}$]



Thm (Eliashberg): (For any singular foliation $\mathcal{F}_0$ divided by Γ_0)

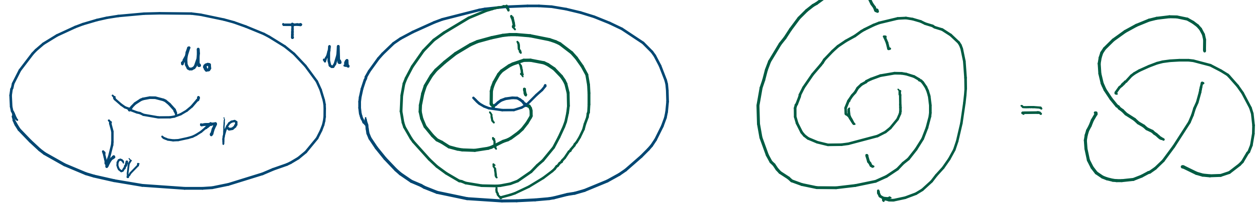
Up to contact isotopy there is a unique tight contact structure on B^3 (printing $\mathcal{F}$ on S^2)

$\Rightarrow$ Any two tight contact structures on $V = S^3 - N(L)$ printing $\mathcal{F}'$ (or $\mathcal{F}$) on $\partial N(L)$ are isotopic.

Take any diffeomorphism $\varphi: S^3 - N(L) \to S^3 - N(L_0)$

and take $\varphi_* \xi_{st}|_{S^3 - N(L)}$ is isotopic to $\xi_{st}|_{S^3 - N(L)}$ $\square$

Torus knots

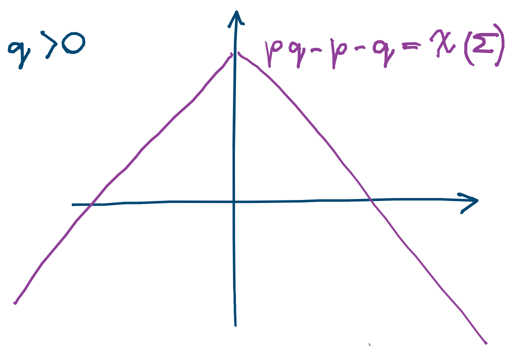


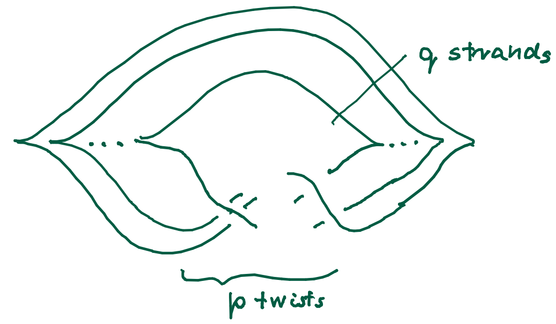
(p,q)-torus knot $(2,3)$ torus knot

$(p,q)=1 \quad q>0$

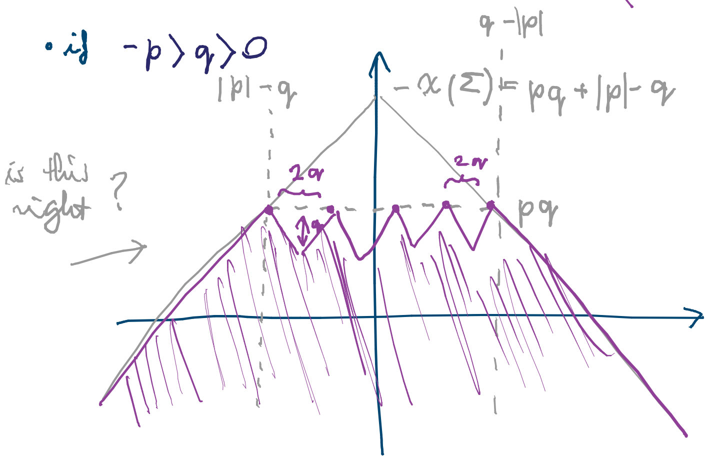
Thm (Etnyre - Honda, 2001): torus knots are Legendrian simple.

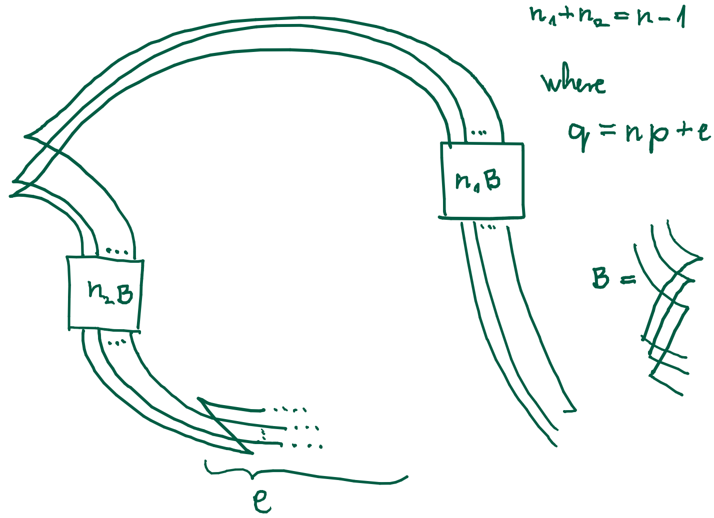
• if $p,q>0$





• if $-p>q>0$





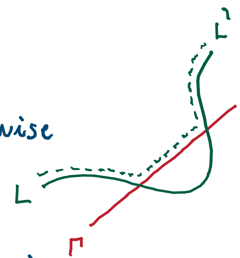
Proof of the $p,q>0$ case: Again, need to prove: if L is a Legendrian representative of $K_{(p,q)}$

A: $tb(L)<pq-p-q \Rightarrow L$ destabilizes

B: $tb(L)=pq-p-q \Rightarrow L$ is Legendrian isotopic to the one given in the Thm

B: 1 $\quad tw(L;T)=\underbrace{tw(L;\Sigma)}_{tb(L)}-pq=-\underset{\vee\atop 0}{p}-\underset{\vee\atop 0}{q}<0 \Rightarrow T$ can be made convex rel L
(Seifert surface)

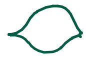
2 Since L does not destabilize $|\Gamma\cap L|$ is minimal ($\lightning$ otherwise L' is nonseparating on $T \overset{LRP}{\Rightarrow} L'$ can be made Legendrian and then L' is a destabilisation of L $\lightning$)

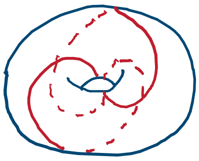


3 Suppose that Γ has $2n$ components of slope $-(r,s)$

Then $tb(L)=pq+\underset{\parallel \atop -\frac{1}{2}|\Gamma_T\cap L|}{tw(L;T)}=pq-n(pr+sq)$ is maximal $\Rightarrow n=1 \quad r=1 \quad s=1$

(HW) Prove that the minimal intersection of a (p,q) & $-(r,s)$ curve is $pr+sq$.

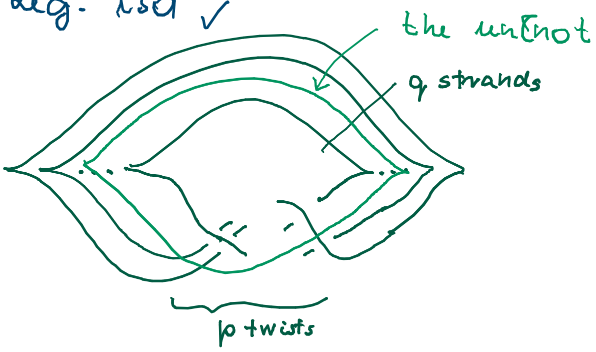
Rmk: U_0 is the standard neighborhood of 



We can assume that T is in standard form and L is one of the ruling curves (which are all Legendrian isotopic)

If L' is another (p,q)-torus knot with maximal tb. Then by the exact same argument L' is a ruling curve on a convex torus in standard form with dividing curves of slope $(1,-1)$ = standard neighborhood of 

Then by the argument in the previous proof there is an isotopy of S^3, ξ_{st} that takes T' to T, and thus L' to a ruling curve of T. Since ruling curves are Legendrian isotopic L & L' are Leg. isot ✓



B.0 We do everything as before to conclude that L is a ruling curve on a convex torus T with $2n$ component dividing curve of slope $-\frac{r}{s}$

→ if $n=1$ & $-\frac{r}{s} \neq -1$. By changing viewpoint (thinking of T as the boundary of U_0 or U_1) we can assume that $-\frac{r}{s} < -1$

Fact: U_0 contains a convex torus with any slope in $[-\frac{r}{s}, 0)$

⇒ 1 contains a torus T' with slope $=-1$

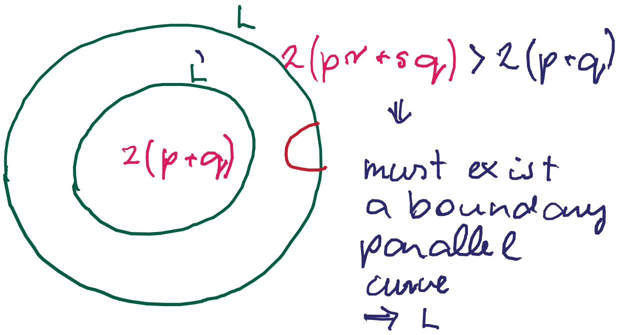
2 isotope T' so that it is in standard form with slope $\frac{p}{q}$ ruling curves

3 take the annulus A bounded by L & and one of the ruling curves L' of T'.

$tw(L,A) = tw(L,T) = -pr-sq < 0$

$tw(L',A) = tw(L',T') = -p-q < 0$

⇒ A can be made convex



⇓

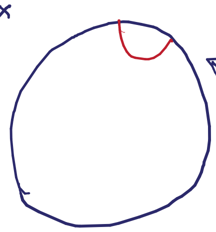
must exist a boundary parallel curve

→ L destabilises

→ if $n > 1$

1 We can decrease n:

take a meridional disc & make it convex



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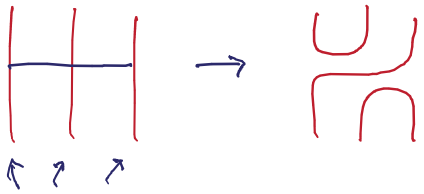

(HW) We get $2(n-1)$ component dividing curve with the same slope

all parallel & coming from different components

& now we can repeat the above argument.

Will not prove the statement for negative torus knots.